INFINITY AS METHOD.

AFTER the mathematical theory of assemblages had been developed, the logic of infinity entered upon a new stage. If we cannot as yet determine the final form of the doctrine, we can, at least, see at the present time in what direction it is tending. After the discovery of the calculus of infinite aggregates and after the establishment of different exactly distinguished kinds of infinity, the perpetual problem as to the potential or actual character of mathematical infinity seems to incline toward a solution in terms of actuality.[1] But this actuality seems to belong rather to the methodological than to the quantitative character of infinite aggregates; *it is a property of methods, not of quantities,* and expreses merely a peculiar system of laws and principles logically working in and upon finite magnitudes. From this point of view infinity cannot be regarded as a kind of quantity, but rather as the *type of structure* of certain quantities;—it does not pass the "limits of our possible experience" as if it were an expression for something "we never could find on sea or land." It is one of the constructive laws of our normal

[1] G. Cantor, "Mitteilungen zur Lehre vom Transfiniten" (*Zeitschrift für Philos. und philosoph. Kritik,* Vol. 91, pp. 81 ff. Comp. Vol. 88, pp. 224 f). Couturat, *De l'infinie mathématique,* pp. x, 213-256; 488-563. Royce, *The World and the Individual,* Vol. II, pp. 554 ff. Spaulding, *Defense of Analysis* (New Realism). H. Lanz, "The Problems of Immortality" (in Russian language, *Logos,* 1913). Gavronsky, *Das Urtheil der Realität* (Dissert., Marburg). Cohen, *Logik der reinen Erkenntniss,* pp. 102 ff. The actual character of infinitely small elements has been mathematically established in Veronese's *Grundlagen der Geometrie von mehreren Dimensionen* (transl. from the Italian). Methodological character of infinity without acceptance of its actuality is emphasized by Brown, *Intelligence and Mathematics* (Creative Intelligence).

experience, perfectly incorporated within every finite object. If two parallel lines do not cross each other in some infinitely distant point, the geometrical structure of our triangles, circles and the like will be different. This mysterious "point" has an *influence* upon finite geometrical forms which is somehow manifested in their inner relations. But is it the "point" that makes the relations what they are? Why not invert our presumption? Why not suppose the structure of finite relations to be prior, infinity being a result of them?

If we can discover all the methodological principles that have to do with infinity, and investigate the reflection of "endlessness" into the world of finite magnitudes, we shall be led to the conclusion that infinity is only an aspect of our normal experience, a property of finite things; that the infiniteness of "space" is nothing but a character of "single spaces," the infinity of series only a property of certain magnitudes. Mathematically however every "property" is an expression of some constructive method. Accordingly the old question as to whether infinity is a category of the qualitative or the quantitative sort, appears unexpectedly in a new light: it seems to be prior to both sorts, being a complex of methods rather than a *quale* or even *quantum*. Those peculiar "qualities" which belong as much to the spatial point as to the instant of time, as much to the sum total of algebraic numbers as to every general concept in its infinite integrating function, lose thus their metaphysical mystery. A point without determined "position" has no peculiar "quale"; taken abstractly it is pure nonsense. This mystical "quale" properly belongs to a point only in so far as it is a *point of a curve. It is a moment in the continuous change of direction,* expressed in the methodical exactitude of a derivative function, not a mysterious "part" of space without extension. For mathematically infinity is always connected with exactly deter-

mined methods of operation, in principle different from addition or division, even in their indefiniteness. This difference in the methods of operation makes every instance of mathematical infinity actual in spite of the impossibility of its production by any arbitrary long process of addition or division. The method of integrals, for instance, perfectly achieves a task which the ordinary methods of addition and division can never fulfil: it leads us to apparent "qualitative" new results,—sometimes to new kinds and unexpected generalizations of number.

The matter in question has often been discussed in mathematical and philosophical literature; but it may be of interest to follow the development of the methodical meaning of infinity as well in the deepest metaphysical speculations of antiquity as in the exactest mathematical achievements of our own epoch. We shall find, then, that this aspect, consciously or unconsciously, is predominant in every "case" and every "kind" of infinity. Thus will become clearer and clearer that positive moment, implied in the earliest unmethodical negations of finiteness, and the "dark" quality as well as the unintelligible "endlessness" of the infinite will be seen to be but concentrated expressions for certain methods of operation involved. This point of view may, possibly, remove the usual distrust and disgust of mathematicians for the "actual infinite."

Qualitative infinity. The infinite is not "not-finite." From the first historical use of the term by Anaximander, the ἄπειρον purports to be a *positive* principle for the explanation of all single and separate "quales," being far from an equivalent for pure nothingness or endlessness. As an ἀρχή or first principle of being, τὸ ἄπειρον is "infinite in a positive sense because it expresses a belief in and a demand for a "different nature" (ἑτέραν τινὰ φύσιν) required for the genesis and derivation of "beings" (ὄντα) —a kind of primitive generating ἕτερον (ἕτερόν τι τῶν

στοιχείων). To be principle is essential for the ἄπειρον; it is nothing but principle or basis (ὑποκείμενον) and has no meaning apart from its "consequences," innate or implicit in it.[2] The *infinitum* is possible and intelligible only in its relation to finite objects, in its logical activity as constructive and productive principle of finite results. It is prior to everything, because τὸ ἀόριστον πρὶν ὁρισθῆναι,[3] and it cannot be constructed by the successive addition or condensation of all things (μεταβολὴ τῆς ὕλης) because its nature is positive, simple and indivisible.[4]

These vague speculations of the earliest Greek philosophy do little more than mark out the field for later analysis. Nevertheless they clearly indicate the primary phase of the problem, in which the finite being begins to look for its origin beyond itself—*in infinitum.* The non-finite, that which is to explain the finites, defines itself as the *problem* of "in-finite." The dim historical previsions, concealed in this definition, soon reveal their positive purport. Aristotle—that scholastic lover of subtle and sterile distinctions—in our present problem brought out a discrimination of great importance. He set up two different concepts of infinity which might have been of a great historical influence and systematic fruitfulness. In his Metaphysics we meet with the clear distinction between the potential (τὸ δυνάμει ἄπειρον) and the actual infinity (ἐνεργείᾳ ἄπειρον). The potential infinite persistently remains within the confines of finite processes and means nothing but the possibility of continuing these processes indefinitely. It remains in the methodical power of the measure, in the

[2] Ἁμαρτάνει οὖν, τὴν μὲν ὕλην ἀποφαινόμενος, τὸ δὲ ποιοῦν αἴτιον ἀναιρῶν· τὸ γὰρ ἄπειρον οὐδὲν ἄλλο....ἐστιν. *Anaximander Milesius* (Neuhaeuser, p. 6). Comp. Simplicius, *Phys.*, p. 32, β: ἐνούσας γὰρ τὰς ἐναντιότητας ἐν τῷ ὑποκειμένῳ ἀπείρῳ. Therefor the ἄπειρον being a different nature, it is not apart from reality. These sentences of Anaximander may be the first indication of the latter metaphysical speculations of Fichte and Schelling, according to which doctrine the "principle" without appearance is nothing; it appears necessarily and exists only in its manifestation in the world of finites; comp. the author's "Fichte and his Doctrine of the Absolute" in Russian).

[3] Aristotle, Met. I, 8, p. 989. [4] Neuhaeuser, *Anaximander Milesius,* p. 44.

field of its logical activity; it is adjusted for measurement and adapted to its "being gone through," but cannot be "gone through" because of the absence of the end.[5] This kind of infinity is only possible as a finite *variable* quantity in process of augmentation or diminution, something remaining always "beyond."[6] This purely negative concept of endlessness, unintelligible and contradictory as it is, has been generally acknowledged by mathematicians and is still current in that branch of science. The other logical type or variant of infinity given by Aristotle is without mathematical import or value and was meant to satisfy the purely logical interest in the notion of οὐσία. This ἐνεργείᾳ ἄπειρον (*infinitum actu*) is apparently the historical source of the "qualitative infinite"—an infinite transcending the problem, the function and the methods of measurement, and as remote from any implication of process as melody is inaccessible for sight. This kind of infinity *being beyond the concept of measure* is thus without "extension"; it has no "middle," no "above," no "below";[7] it does not consist of "parts" and is in the strict sense indivisible.[8] We must give over enumeration if we want to grasp infinity *in actu*; it is impossible to understand or to construct it in terms of continued recurrence of finite elements; in a word actuality marks in this primitive stage the creation of a new quality, the elevation of the mind to an entirely different level expressible only in terms of "ideality." Historically the meaning of ideality is connected inseparably with infinity[9] because to consider any fragment of reality under the aspect of ideality means to consider it as an instance of universal conformity to law. "Ideality" is the explanation of infinity *in actu,* or the resulting "quale" of infinity.[10]

This new qualification has been for many centuries the

[5] Aristotle, Met., κ, 10. [6] Aristotle, Phys., γ, 6. [7] Aristotle, Phys., γ, 5.
[8] Aristotle, Met., κ, 10. [9] Comp. Schelling, *Bruno, S. W.*, I, 4, pp. 342 ff.
[10] Hegel, *Logik, S. W.*, III, pp. 165, 171 ff.

main subject of philosophical reflection. In terms of infinity Plotinus defines his overtemporal realm of creative intelligence, where every "part" possesses the same "power" as the whole.[11] In the same terms Spinoza constructs his concept of substance, by which the must have meant to express neither more nor less than the logical inevitability of all the laws of nature.[12] In medieval literature also we meet with a very instructive instance of the qualitative infinity. I mean the doctrine of eternity. Thus Anselm constructs his concept of God in terms of "eternal truth" by the method of time-negation;[13] and this eternity is no potential or repetitive infinity of time. It transcends all lapse of time; it is an "indivisible unity" beyond time,[14]—*tota sibi praesens.*[15] This is the original source from which the modern concept of *Geltung* or Ideality has been derived. And what is of more importance, Anselm in his explanation of "over-temporality" goes further perhaps than Bolzano, Husserl, or even Bradley. According to his doctrine, the irrelevance to time limits not only produces a peculiar quality but is caused by negation and suspension of all the methodical means used for the explanation of temporal reality. *The positive ground for this new quality is discovered in the conformity to a new law.*[16] which has found its positive expression and justification in the laws of Logic.

Thus the definition of infinity as "quale" reveals its positive value when applied to the problems of pure logic.

[11] Plotinus, Enneades, III, 8, 8; VI, 9, 6. Comp. Henry Lanz, "Speculative Transcendentalism in Plotinus" (printed in Russian in the Journal of Ministry of National Education, 1914, I, 2.

[12] Comp. Wenzel, *Die Weltanschauung Spinozas.*

[13] Sancti Anselmi opera omnia (*Patrologiae cursus completus* T., 159), Monologium, pp. 160, 198 ff; Proslogion, pp. 235, 237, 239; Dialogus de veritate, p. 479.

[14] Anselm, Proslogion, p. 237.

[15] *Ibid.*, p. 238.

[16] Anselm, *Monologium*, p. 175: "Procul dubio summa substantia, quae nulla loci vel temporis continentia cingitur, *nulla eorum lege constringitur.*" Comp. *ibid.*, p. 166: "Ita uno modo, *una consideratione* est, quicquid est essentialiter." Comp. *ibid.*, 184-185. Thus according to Anselm's conception the essentia or idea is not apart from reality but a certain "consideration" of it, the result of the methodical application of certain laws.

Every logical content, every proposition in its value and relative truth can be regarded as an *instance of infinity,* because it marks the abandonment of the primitive attitude of enumerating the single cases—a turning from the sheer pluralism of sense-perception to the universality of method established first in Plato's "idea." The result was a different kind of logical operation, absorbing in the *process of deduction* all the possible cases *in infinitum,* instead of their enumeration *in indefinitum. The infinity implied in every logical concept is "actual" not because all its single cases have been enumerated, but because enumeration is no longer significant or serviceable.* In this sense "actual" infinity is simply the expression of a generality implied and employed in all use of "general concepts." It is no inherent or peculiar quality of "ideas" enabling us to apply to them our deductive, dialectical or transcendental methods; on the contrary by our methods we fashion our "ideas" into a logical form and adaptability in which they have for us the semblance of "transcendent entities."

Our modern logic is a positive system of methods, laws and categories which has grown out of these metaphysical speculations concerning eternal ideas, substances, God, *absoluta* and the like. It is an historical outcome of the simple resolve to consider the separate cases not in their plurality but in their systematic totality. So considered, they inevitably stand revealed as infinite logical complexes. Their being instances of a qualitative infinity is nothing but the expression of what they are as instances of a methodical (in a large sense deductive) thinking, and the peculiar quality of logical concepts, expressed in their eternality, overtemporality and so on, is nothing but a peculiar kind of operation with such complexes which justifies them and gives them a definite meaning. They have no existence for us until our methods of dealing with our world have made them seem to exist. Whereupon we say metaphys-

ically, "they exist in no space or time," "they have but an intentional being." The peculiar property or power of logical complexes to embrace an infinity of single cases compels no recognition of a mysterious realm of truths or ideas, beyond reality. It is a natural consequence of our methodological emphasis upon the significant proportions and relations of the finite cases ("The essence is immanent to the appearance,"[17] "objectivity is created by consciousness," "the mind prescribes its law to nature").

Critics of the potential infinity. The usual explanation of infinity in mathematics consists in the assertion that the true infinite is nothing but a symbol expressing the possibility of continued counting or measurement—in a word "potential" infinity.

We may urge against this "subjective" principle of explanation the general objection, that our process of counting does not belong to the objective value or logical purport of any mathematical relations,—it can explain nothing, it can prove nothing with regard to them: it is absolutely irrelevant to their logical constitution or value, and can play no part in their mathematical establishment. The relations which govern in this operation, as executed by our mind, are psychological or epistemological relations which can have nothing to do with arithmetic or with the theory of aggregates. The constructive principle or method of arithmetic excludes by its essential purport all influence of consciousness so that what is impossible for consciousness may be quite possible in principle. This *elimination*

[17] The "essence" by Spinoza—perhaps the most important instance of the qualitative infinity—is based on the same methodological ground; to regard anything in the essence, as a modus of substance, means to regard it ***sub specie aeternitatis, as a moment in the deductive development of the system;*** everything *is* "substance" in so far as it is an instance of "method," i. e., in so far as it has truth (***sub specie veritatis***) and can be proved. The "essence" does not point out a different thing provided with special qualities, side by side with its real appearance, but the essential relation *within* the appearance itself, its conformity to the immanent law, ***its ability of being proved, its position in the system of deduction.***

of consciousness belongs in fact to the logical intention of all theory as such and is of the essence of all reasoning.[18]

This general proposition has an application to our present discussion. To base the concept of infinity on the ground of mental possibilities or processes means the same as to construct it without any ground, because the recurrence to the process as such has no logical value; the whole reasoning represents a very simple example of *quaternio terminorum.* What properly plays a part in the constructive definition of infinity is not the process as such (as executed in a finite time), but the inner methodical principle of it;[19] the process itself is going on according to this principle, changing no element or item of its logical content or axiomatical formulation. It would be absolutely meaningless to say: the principle does not define the totality of a certain series, because we are unable to stop in our process (*quaternio terminorum*). On the contrary: we cannot rest in our process because the principle does not permit us to rest, because it gives no guidance or indication as to a particular point of absolute rest. The relation must be reversed: it is the logical nature of infiniteness which gives our processes of counting or measuring indefinitely large or small, not contrariwise. Thus the *infinitum potentiale* may be called groundless infinity; the process of its construction is based on a well-known logical fallacy.

18 The general position here indicated has been elsewhere worked out by the author. Comp. "Das Problem der Gegenständlichkeit in der modernen Logik" (Ergänzungsheft d. Kantstudien, No. 26, 1912); "Fichte und der transcendentale Wahrheitsbegriff." It has also been developed in certain writings of the author in the Russian language (in "Logos" and "Problems of Philosophy and Psychology").

19 From this point of view may be profitably discussed a very old doubt of the sceptics, that for the purpose to know the infinite "we" must have an infinite capacity, and since we do not have any, we are unable to know anything about infinity. Pascal says, for example: "....il ne faut pas moins de capacité pour aller jusqu'au néant que jusqu'au tout; il la faut infinie pour l'un et l'autre." (*Pensées*, I, p. 82. Nouvelle éd. par Brunschvicg). But as we have seen, any of our "capacity" does not belong and does not have any logical influence upon the content known; therefore we don't need to have an infinite

On the other hand, the mathematical possibility of setting up a limitation in counting ("to construct the concept of number") is dependent upon a long series of special, and only for that purpose inevitable, pre-suppositions, which shall axiomatically define the meaning of the end. After the infinite aggregates had been mathematically defined by means of equivalence between part and whole, a long series of presuppositions was required to construct the finite numbers. The principles of enumeration, established in Dedekind's system of arithmetical axioms by means of "similar representation" and the concept of "chain," are similarly limitations added to Cantor's "axiomatic." Every aggregate of elements may be a "number" not by itself, but only in so far as a certain principle of representation is used; the same aggregate might be ordered by some other principle in a different way, which does not define the fundamental laws of finite arithmetic.[20] Thus the system of axioms which define the transfinite aggregates is prior to the system which defines the finite numbers (in metaphysical language: "finite things are limitations of infinite," ἀόριστον πρὶν ὁρισθῆναι); that is to say: the actual infinite is presupposed by the potential infinity.

Infinity of Series. If the subjective ability to continue a certain kind of operation *in indefinitum* does not belong to the constructive value of infinity, then what is the meaning of this peculiar term? It has to be determined without any reference to subjectivity—otherwise it would have no meaning at all. Let us start with the consideration of infinite series.

We may express the approximate value of π in decimal notation.

$$\pi = 3.14159 \cdots$$

mind for the purpose to grasp the infinity *in actu.* The same might be said against Kant's doctrine.

20 Dedekind, *Was sind und was sollen die Zahlen?* p. 37.

It is plain there is no sense in speaking forthwith about the whole decimal series in this expression, simply because the terms of the series are not represented by it. The equation gives no enunciation of what numbers are to be united into a whole; however long we may continue the enumeration of decimal terms, the value of π still remains under the curse of the undeveloped "potentiality." Nevertheless, π has an exact geometrical value, being a symbolic expression of a certain type of relation which cannot be expressed by any other value.[21] Otherwise, *every* decimal in the above series not only can be but objectively *is* determined—whether we compute it or not—by a certain method of operation, *each* of them *is* a result of exactly defined proportions and relations between the finite numbers. The theorem of Taylor supplied the foundation upon which have been based different methods for estimation of the value of π. Suppose we have carried out the calculation of π until we have reached the 707th decimal; we ask: are the 708th or 1000th decimals objectively undetermined? Would they be "created" in the process of our further calculation and begin to be only in that moment when we *know* their value? However it may be with the question of the dependence or independence of being upon consciousness, it is evident enough that the logical value of a certain mathematical proposition does not begin to "exist" with our temporal act of knowing it; all the roots of a certain algebraic equation of nth degree, for example, have their mathematical "existence," i. e., they are perfectly determined, in spite of the fact that we are forever unable to know them. How much more right then we have to conclude that *every* decimal in the objective value of π is determined by a certain type of preserial relations (theo-

[21] Comp. Couturat, *De l'infinie mathématique,* pp. 216-217: "En admettant que le symbole $m/0$ n'ait pas de sens numérique, il ne laisse pas d'en avoir un parfaitement intelligible en géométrie: car ce qui est absurde ou illégitime au point de vue du nombre pur ne l'est plus au point de vue géométrique." Comp. pp. 257; 213-256; 488-503.

rem of "middle worth"); and we are able to continue our calculation indefinitely only *because* every term—whether we actually know it or not—is objectively determined. Not in the process, but in the method of this determination our series becomes infinite. We should have no right to speak of the infinity (not even the potential infinity) of our series were there no *law* generating all the terms of the series. Therefore the working principle and real essence of infiniteness in our present case is represented by this generating law which renders it actual. Mathematically the infinite series may be regarded as given in its totality ("the series defines an infinite number")[22] only when its "general term" is given, because the constructive law is then expressed immediately in the form of series:

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\cdots\pm\frac{1}{2n-1}\pm\cdots$$

Of course the infinity in this case would be meaningless if we tried to construct it by the successive addition of the terms; because this addition can never be fulfilled and the series does not mean anything else but the virtual determination of "every" term by a certain law; and the indefiniteness of series is nothing but a reflex of the actual infinity of law, being a negative expression of its positive character. Thus the infinity, being an instrument or methodical concept rather than one of process, is free of every continuation; it cannot be constructed by the methodical means of continuation, because the process as such is irrelevant and does not belong to the purport of infinity; it is rather an expression of the structure of certain processes than a concrete process by itself.

The reproach that we are unable to accomplish the measurement of a circle whose diameter is equal to one, does not prove the impossibility of such a circle; its circum-

[22] This terminus: infinite number is introduced by Dingler to denominate every mathematical expression, containing infinite chain of operations and united by a certain and expressible law.

ference possesses its exactly determined value *in the constructive law* of π; this transcendent value is *not approximately but exactly* determined by this law, expressed by its "general term" or more deeply by Taylor's general formula. We don't need to estimate all the terms for the purpose of determination of series, which is sufficiently determined independently upon the estimation of all approximative value in the series.[28] It is absolutely wrong to suppose that the mathematical determination is possible only by the arithmetical calculus; it can be afforded also by the simple indication of the constructive law. The arithmetical determination represents only a particular case of this general rule, every "number" is but a symbolic expression of a certain "type of order"); it is not a self-contained entity separated from the concrete reality but rather a peculiar way of bringing order into the concrete world of experience, a method of operation. The mathematical "existence" of the various kinds of numbers signifies only the possibility of determining magnitudes by their constructive law; "existence" means nothing but determinateness of whatever sort. The number 2 is not less determined when not defined by the constructive serial law (general term)

$$2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdot + \frac{1}{2^n} + \cdots$$

than by its arithmetical definition:

$$2 + 1 - 1$$

Therefore the number 2 may be an "infinite number" as well as π; the difference is only a formal one, the determination of irrational number by infinite series may be used as their definition. This is impossible in reference to the rationals only, because it would be an apparent circle

[28] Comp. Euler, *Vollständige Anleitung zur Algebra,* Ges. Werke, Vol. I, pp. 50-51. He says: "One cannot say, that $\sqrt{12}$ is in itself undetermined, but from what has been just said it follows only that $\sqrt{12}$ cannot be expressed *by fractions,* nevertheless it necessarily has a determined value."

in definition as may be seen from the equation given above. Returning to our previous example, π is sufficiently distinguished from every other value by the simple character of its constructive law and can be used as a well-defined complex of numerical relations;[24] its value is an actual one, and we have the right to regard it as actual only because it can be expressed in the terms of series having on infinite law. Thus *infinity is rather a property of law than of series as such.*

As to the character of the constructive law in general, another limitation has to be made. It is evident that our reasoning has no application to the divergent series. Of course, the law of a divergent series may be actual too, but it is meaningless to call it constructive, because it does not construct anything. The infiniteness in this case loses its purpose and that may be so important as to lose ground. For infiniteness is completely expressed only in and by the fulfilment of a certain task; where no determinate task is set, no fulfilment is possible. This might have made the concept of "limit" of such a great logical importance. But, from the logical point of view the limit is a statement of problem rather than a solution of it; the "existence" of limit is proved by the constructive law of series, not contrariwise, and the peculiar "jump" made by our mind in transition to the limit remains still unexplained and unintelligible if the methodical sense of actual infinity is disregarded. The infinite constructive law leads us to a determinate, mathematically positive result, i. e., it justifies the establishment of limit only when the series satisfies the conditions of convergence; that is to say: "the *infinitum* is possible and intelligible only in its relation to finite magnitudes, in its logical activity as constructing and pro-

[24] Every "infinite number," like π, may be regarded as an instance of such a "whole" which contains more than *all* its "parts"; because every "part" or "cut" of the series is a rational number (according to the definition) and the "whole" leaves the limits of rationality in principle, breaking the methodical law of it.

ducing the finite result" (metaphysically: "to be principle is essential for the ἄπειρον," "*das Prinzip erscheint nothwendig*").

The following important conclusions may be derived from the analysis of series given above: (1) Infinity has its logical basis in the constructive law (positive) which *makes* the series endless (negative). (2) The possibility of continuing a certain process indefinitely is nothing but a reflex of the objective activity of certain laws (not every methodical law gives this possibility). (3) The purport of infinity is justified only if it gives and implies a method for the genetic creation of the finite (conditions of convergence). (4) The qualitative moment in the logical explication of infinity turns out to be the methodical efficiency and defines itself more precisely as the logical activity of a certain type of laws.

The infinitely large (transfinite) numbers.[25] We are brought to the same conclusions by the consideration of transfinite numbers. What did Cantor mean by his "actual infinity"?

If we compare the whole lot of algebraic numbers with the series of integers, no conclusions can be immediately derived as to the quantitative difference between the two assemblages; there is no sense in speaking of the totality of an indefinite fraction, in default of a method for its construction. To Cantor chiefly is due the credit of preparing the way for the methodical comparison and mathematical treatment of such indefinitely large multitudes. Instead of classifying "numbers" by themselves he classifies the roots of *all* the algebraic equations, and in doing so he makes it possible to insert them in a proper order in which every equation finds its "enumerable" place in accordance

[25] This paragraph I suppose to be in agreement with Pr. Brown in his remarkable article: "Intelligence and Mathematics" (*Creative Intelligence*) in which is emphasized the methodical conception of transfinite numbers. But I cannot agree with the author concerning the potential character of this type of "numbers."

with its "height"; the classification of real roots follows immeditely and of itself because every equation has a finite number of roots which may be disposed in a proper order in accordance with their magnitudes. By this simple method of disposition he was able to coordinate all roots and consequently all algebraic numbers to the series of integers; obviously in the process of this coordination, not one root remains without a "number."[26] This method of "univocal coordination" first opens to us the logical possibility of applying the concept of "whole" to such indefinite aggregates, "wholeness" (infinity) signifying nothing but the unrestricted action of a certain law of coordination within certain limits. The aggregate of all the algebraic numbers can be regarded as a "whole" (transfinite number) only because *it can be arranged in the same way as the series of integers is arranged*; this prior constructive principle makes both classes of numbers equivalent, in spite of their indefiniteness rendering them of the same "class" and endowing them with the same "power." Every different "power," i. e., every *aleph*, means a different way or arrangement rather than a new (larger) quantity; by the general method of "covering" (*Belegung*) we are able to produce new and newer kinds of infinity, i. e., ever new ways of arrangement. Thus here also infinity is perfectly imprisoned in the finite magnitudes and expresses nothing but a peculiar method of organization of our usual experience.

The mysterious equivalence between "whole" and "part" loses its paradoxical character if we consider the situation from this methodical point of view. *It does not express equivalence in* quantis, *but only an equivalence in* methodis. The whole lot of algebraic numbers quantitatively is the same aggregate as the whole lot of integers, but differ-

[26] Cantor, "Ueber eine Eigenschaft aller realen algebraischen Zahlen (*Journal für reine und angew. Mathematik,* Vol. 77, pp. 258-268).

ently arranged; the "addition" of new terms does not change anything in this arrangement, which *methodically* remains the same, i. e., does not increase it at all, just as the designation of new officials does not increase the population. Cantor categorically distinguishes the "logical function" by which the transfinite numbers are established and proved, from the method of successive addition of terms.[27] The transfinite number, consequently, does not "consist" of its parts, because it is not constructed by regular addition. To take away a proper part from a certain transfinite aggregate does not change anything in it, *because the part has never been added.*

The infinitely small. The usual protest of mathematicians against the actual character of infinitely small elements has its ground in the tendency to understand it in terms of extensive magnitudes, i. e., infinitely small *parts.* Of course under such an aspect the concept of the differential becomes impossible and even contradictory. But still the infinitely small may be regarded as actual from the standpoint of such categories as lie beyond the competency of the primitive opposition between part and whole; this opposition may be irrelevant in the process of its logical construction. For differentiation is not division, and the mathematical procedure of this operation has no resemblance to the process of division. It is logically impossible to establish the concept of an indivisible part.[28]

The task of differentiation according to Leibniz is not to find infinitely small parts; the differentiation has to do with laws, i. e., functions, instead of with extensive magnitudes. If the constructive law of a certain line is given, we determine by differentiation, not a "point," but the direction of the tangent in this point; if a certain law

[27] Cantor: *Grundlage einer allgemeinen Mannigfaltigkeitslehre,* No. 11.

[28] Leibniz, *Mat.* III, 524: "Etsi enim concedam, nullam esse portionem materiae quae non actu sit secta, non tamen ideo devenitur ad elementa insectabilia aut ad minores portiones, imo nec ad infinite parvas."

of movement is given, differentiation determines the velocity at every moment of this process. Thus differentiation is not an algebraic action with extensive magnitudes but a way of dealing with laws or functions as such: when a certain law is given, differentiation permits us to determine the action of this law at every moment and for every element, i. e., to find the derivative function. On the contrary, if the derivative function is given, i. e., the action of a certain law at every moment is known, we are able by the inverse operation of integration to determine the law itself (the equation of a certain curve for instance), i. e., to find the original function (*data lege declivitatum curvae, posse describere curvam*). Within this operation the curve is regarded not as an aggregate of points, not as what it "consists of," but as a continuous change of direction; in the same way we are to conceive movement not as a product of "rests" but as a continuous change of velocity. The well-known paradox of Zeno is a simple case of fallacy; from the fact that in the 0 of time a point traverses no space, nothing follows as to the point's velocity. The expression $0/0$ is arithmetically undetermined and undeterminable; but Zeno fallaciously ascribes to this point a determined velocity—rest being a peculiar case of velocity, where $v = 0$.

The differential does not exist as a part of the integral; under the aspect of part and whole (arithmetically) it is a pure nothingness; it has a meaning only as an instrument for certain operations with magnitudes, not with numbers. From this numerical point of view 0 is an expression for the simple negation, or absence of being: $0/0$ here has no sense and must be regarded as an absolutely undetermined form, precluding all mathematical treatment. In the series of magnitudes the zeroes do not exist at all; here the concept of zero must be supplanted by the concept of "moment" which always conserves its specific qualitative

character. A point in space, an instant of time and a moment in a movement are qualitatively distinguished in spite of the absence of any quantitative element in them. But the concept of this peculiar quality again is a statement of the problem rather than a solution of it; the quality has no mathematical expression and cannot be used in the process of calculation. What may be regarded as conserved is not the quality but an exactly determinable relation which is active in every moment of the process. In this sense the point of a circle is different from the point on a parabola, because each keeps the direction continually produced by all other points of its "class"; every element of a curve implies the law of its curvature; every moment of the movement continuously keeps the law of its velocity. Therefore the point, in so far as it is an element in the continuous change of direction, is not a simple null, but such a null as logically to imply the law of the whole line, i. e., the "infinitely small element" of "differential."[29] This infinitely small element has a meaning only in reference to a corresponding magnitude, determined in its whole character by a determinate formula or law (*interventu infiniti finitum determinatur*).[30] Another account of the infinitely small is possible. According to Leibniz's principles the differential has no positive meaning in itself; it seems to be only a symbolical expression of what is logically tctive *behind* it in the methodical meaning of the derivative function $(dx:dy)$.[31] Is it possible to ascribe any positive value to dx independently of dy? Veronese tries to solve this problem geometrically. He defines the infinitely small "segment" of the order m as follows: "If a number in relation to another number is infinitely large, then this second number in relation to the first may be called infinitely

[29] Leibniz, *Opera Mathem.*, IV, 218: "Interea infinite parva concipimus non ut nihila simpliciter, sed ut nihila respectiva....id est ut evanescentia quidem in nihilum, retinentia tamen characterem ejus quod evanescet."

[30] *Ibid.*, VII, 53.

[31] Comp. also Euler, *De calculo integrali*, I, 2.

small."[82] In a special theorem he emphasizes the actual character of elements so defined, exactly distinguishing them from the "indefinitely small" elements: the latter still remain in the process of diminution and belong to the series of finite magnitudes; but it would be meaningless to seek the infinitely small segments in the series.[83] Bernoulli already believed the actual existence of infinitely small elements: "Sic omnes termini hujus progressionis, ½, ¼, ⅛, 1/16,.... actu existunt, ergo existit infinitesimus....si decem sunt termini existit utique decimus, si centum sunt termini, existit utique centesimus...ergo si numero infinito sunt termini, existit infinitesimus." But Bernoulli still seems to believe that the *terminus infinitesimus* is given in the series itself; of course, were this true, Veronese says, the concept of the infinitely small would involve a contradiction, because all the terms of this series according to the definition are finite. But it is logically possible to regard the series as defining a certain element beyond itself, which does not belong to the class of numbers given in the series, being always smaller than every term of it; and nevertheless this element may have some exactly determined and definable properties. In a certain circle of problems the assumption of such element may even be inevitable. Suppose we hypothetically accept the assumption of such a system, where "every finite segment, variable as to length and becoming indefinitely small, contains an element, which is different from its terminal points"; in the first place this presupposition is logically possible and implies the definition of infinitely small elements given above; in the second place it is obvious that certain systems (for instance the system of the spatial points) satisfy this hypothetically accepted condition.

From what has just been said it seems to follow that

[82] Veronese, *Grundzüge der Geometrie von mehreren Dimensionen*, p. 116.
[83] *Ibid.*, p. 141.

the infinitely small may have mathematical existence by itself. But still it remains true that all the determinations ascribed to this element are, properly speaking, of a derivative nature. What Veronese might have had in mind is only a convenient way of expressing certain properties and proportions in a certain class of systems called continuous. He explains himself clearly in this way.[34] If the distance between the two foci of the ellipse remains only indefinitely small, without any suggestion of an element of a different order transcending the potential series of such indefinitely small distances, the circle might never be considered as a particular case of ellipse. The "actual existence" of an infinitely small distance in this case means nothing but the possibility of passing from the formula given for the ellipse to that for the circle. The "actual existence" has here no meaning beyond the methodical value of certain operations; and this methodical value of the "infinitely small" element consists in what it is doing in the system, rather than in what it is. The correlation between the finite distance and the infinitely small element defined by the process of its continuous determination, is no relation between quantities, but in our present case, the relation of affinity between two different laws (circle—ellipse). Their "truth" consists in what stands behind their formal definition, in the methodological back-ground of their "existence." Still in terms of our present example, we may say that the point (as infinitely small distance) can be regarded as a "part" of the line only because a certain class of analytic forms (circles) can be regarded as a "part" of another more general class of forms (ellipses). It is obvious that the problem here again harks back to the problem of qualitative infinity.

Résumé. From what has been said it follows that infinity in all the cases of its application has a purely method-

[34] *Ibid.*, p. 144.

ical value. It is not a "thing in itself," not something ready given and self-existent independent from science; it is not a "thing" of whatever sort; it is a method, rather a methodical aspect of reality than reality itself. I don't want to allege that it is a pure product of our mind, unless we understand this term "mind" in a purely logical sense, as a system of methodical presuppositions of science, action or art. Then and only then, in this exactly restricted sense of "our mind," it may be logically created by it, i. e., every instance of infinity may be and, as a matter of fact, is a result of certain presuppositions. The reproach of artificiality does not affect our position in any way; in this broad and vague sense everything may be called artificial; I don't see any reason why any finite magnitude or any limited field of experience is less artificial than a transfinite number. We are too much inclined to forget, that a long period first of biological adaptation and then of logical and mathematical reasoning were required to perceive the limits of the real objects and to conceive the meaning of the "end." It is an old truth that all the boundaries in this world are artificial, i. e., they are based upon a long system of presuppositions. But since these presuppositions are not artificial at all, since they have their meaning and purpose, the result of their logical activity loses its artificial character also. Thus to persist in the thesis that everything in our world of experience is limited, is in itself a logical limitedness: It must be considered as a modern positivistic extreme, as a reaction against the metaphysical exaggeration of the value of infinity. The opposition of finite and infinite is not a contraposition of the different realms or worlds separated from each other; it is only a cooperation of two different methods one of which is quite as justified as the other.

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